A STOCHASTIC DIFFERENTIAL EQUATION MODEL FOR THE SHORELINE EVOLUTION

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ABSTRACT

The long-term shoreline evolution due to longshore sediment transport gradients is one of the most important problems in coastal engineering design and management. It is well known that the primary physical factors controlling the shoreline evolution are the wave climate, existing shoreline position, sediment supplies and properties, and the influences of coastal protection structures. Since most of these factors cannot be determined precisely, the dynamical response of the shoreline over time has to be treated as a time-dependent stochastic system. This paper presents a stochastic differential equation model based on the standard one-line model description of the processes for predicting the probability distributions of long-term shoreline positions. The model is applied to a simple coastal configuration involving a single long jetty perpendicular to a straight shoreline in order to evaluate the usefulness of the model.

Keywords: shoreline erosion, long-shore transport, cross-shore transport, wave distribution, Fokker-Planck equation

1 INTRODUCTION

Shoreline evolution is a complex process which is affected not only by the prevailing hydrodynamic forcing but also by the existing morphological conditions of the shoreline. The long-term shoreline changes can exhibit considerable variability which can not be fully predicted by deterministic models. A more rational theoretical framework is to view the dynamical response of the shoreline over time as a time-dependent stochastic system.

In the past decade various probabilistic methods have been proposed for predicting shoreline evolution. Some of these methods dealt with the variability of hydrodynamic input while others focused on the variability in model parameters. Nearly all probabilistic shoreline evolution models are based wholly or in part on the one-line model originally proposed by Pelnard-Considere (1956) and subsequently extended by many other researchers. A notable earlier probabilistic shoreline evolution modelling work is the one reported by Vrijling and Mejer (1992) who performed both simple risk analysis and full Monte Carlo simulations of the shoreline positions due to long-shore transport gradient. In order to account for the influence of cross-shore transport on the shoreline position so as to determine the true risks of shoreline erosion over long-term Dong and Chen (1999) extended this modelling approach to include random temporal variability and storm beach profile changes due to cross-shore sediment transport. Both works have showed that simulation using Monte Carlo methods is applicable to any general shoreline systems but requires extremely large computational resources, especially for accurately computing the tails of shoreline probability distribution. Recently Reeve & Spivack (2004) developed methods that directly solve for the statistical moments of shoreline position and presented the time-dependent ensemble averaged solutions for given wave climates. The advantage of such an approach is that it avoids the need for computationally intensive Monte-Carlo simulations but the drawback is that it does not give direct information on probability distribution of the shoreline positions.

In this paper, a stochastic differential equation (SDE) model is developed in which the physical process is formulated based on a one-line type model and the random effects are assumed to be characterised by a Wiener process. Under these conditions the probability density function (PDF) of the shoreline position at any given time is found by solving Fokker-Planck (FP) equation.

2 ONE-LINE MODEL

The standard one-line model was developed by Pelnard-Considere (1956). It assumes that at the appropriate time scale of description (several months to many years), the beach profile would move in parallel to itself while maintaining its shape. The equation for the deterministic process that links the temporal changes of shoreline positions with the spatial gradient of the long-shore sand transport rate is

$$\frac{\partial y(x,t)}{\partial t} = -\frac{1}{d_{\mathrm{c}}}\frac{\partial q_{\mathrm{l}}(x,t)}{\partial x} \tag{1}$$

where q_{l} is long-shore sand transport rate; x is space coordinate along the axis parallel to the trend of the shoreline; y is the shoreline position, t is time and d_{c} is the depth of closure. The general expression for q_{l} is given by

$$q_{\mathrm{l}}(x,t) = q_{\mathrm{l0}}(x,t)\sin 2\alpha_{\mathrm{b}}(x,t) \tag{2}$$

where q_{l0} is the amplitude of long-shore transport rate, α_{b} is the breaking wave angle to the local shoreline x given by

$$\alpha_{\mathrm{b}}(x,t) = \alpha_0(x,t) - \arctan\left(\frac{\partial y}{\partial x}\right) \tag{3}$$

where α_0 is the incident breaking wave angle relative to x. In general, variables q_{l0}, α_0 and α_{b} all vary in time and alongshore.

For prediction purpose, an expression for q_{l0} in terms of wave parameters and sediment properties is required and one of the most widely used is the CERC formula which is given by

$$q_{\mathrm{l0}} = (E_{\mathrm{b}}C_{\mathrm{gb}})a_1 \tag{4}$$

where a_1 is a constant given by $a_1 = \dfrac{k_1}{(\rho_{\mathrm{s}}/\rho_{\mathrm{w}} - 1)(1-n)(1.1416)^{2.5}}$, E_{b} and C_{gb} are the wave energy and group velocity evaluated at the breaking point, ρ_{s} is density of sand, ρ_{w} is density of water, n is void ratio of sand. k_1 is an empirical constant. Adopting the linear shallow water wave approximation we have $E_{\mathrm{b}} = \dfrac{1}{8}\rho_{\mathrm{w}} g H_{\mathrm{b}}^2$ in which g is the gravity acceleration and $C_{\mathrm{gb}} = \sqrt{g h_{\mathrm{b}}}$. Further assuming $H_{\mathrm{b}} \approx \gamma h_{\mathrm{b}}$ equation (4) can be rewritten as

$$q_{l0} = H_{\mathrm{b}}^{5/2}\left(\frac{1}{8}\rho_{\mathrm{w}} g^{3/2}\gamma^{1/2}a_1\right) \tag{5}$$

3 THE MODEL BASED ON THE STOCHASTIC DIFFERENTIAL EQUATION (SDE)

From equation (5) it can be seen that when random waves are concerned, the randomness of the longshore transport is entirely due to the randomness in the breaking wave height. Therefore, the general random differential equation corresponding to equation (1) together with the corresponding initial condition can be written as

$$\begin{cases} \dfrac{\mathrm{d}y(x,t)}{\mathrm{d}t} = f(y,x,H_{\mathrm{b}}) \\ y(x,0) = y_0 \end{cases} \tag{6}$$

in which the parameters H_{b} is now a random variable and y_0 is the initial shoreline position which is also a random variable as for natural beaches there is always some uncertainty regarding the initial shoreline position.

According to equation (5) the longshore transport rate is dependent on $H_{\mathrm{b}}^{5/2}$ rather than H_{b}. If we let $H_w = H_{\mathrm{b}}^{5/2}$ be a new wave height parameter the random variation of the breaking wave height can be represented by $H_{\mathrm{w}} = K_{\mathrm{m}} + W(t)$ whose mean value is K_{m} and $W(t)$ is assumed to be a Gaussian white noise process with mean zero and a variance of D which is defined as a fraction of the magnitude of the coefficients as $D = C_v K_{\mathrm{m}}$. Based on the one-line model the stochastic differential equation can now be written as

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = -[K_{\mathrm{m}} + W(t)]\phi(y,t); \qquad W(t) \sim N(0,D) \tag{7}$$

where $\phi(y,t) = \frac{1}{8}\rho_{\mathrm{w}} g^{3/2}\gamma^{1/2}a_1 \frac{\mathrm{d}}{\mathrm{d}x}(\sin 2\alpha_{\mathrm{b}})\frac{1}{d_{\mathrm{c}}}$, called transfer function. It is well known that the evolution of such system is Markovian and the stochastic process model of shoreline position given in Equation (7) can be written in a general form as

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = -\phi(y,t)K_{\mathrm{m}} + G(y,t)W(t) \tag{8}$$

with $G(y,t) = \phi(y,t)$ in this case. It can be seen that Equation (8) is equivalent (see Soong, 1973) to the stochastic Ito equation.

$$\mathrm{d}y(x,t) = \phi(y,t)K_{\mathrm{m}}\mathrm{d}t + G(y,t)\mathrm{d}B(t) \tag{9}$$

where $B(t)$ is the Winner process with

$$E\{\mathrm{d}B(t)\} = 0 \qquad \text{and} \qquad E\{[\mathrm{d}B(t)]^2\} = 2D\mathrm{d}t \tag{10}$$

in which D is the entry of the covariance parameter of the white noise process. Furthermore, the stochastic system described by equation (9) has a transition probability density function, $p(y(t),t)$ of random process, $y(t)$, that satisfies the Fokker-Planck equation (Soong, 1973):

$$\frac{\partial p(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left[\phi(y,t)K_{\mathrm{m}}\,p\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[(GDG^T)\,p\right] \tag{11}$$

Therefore the problem of determining the probability density $p(y(t),t)$ is transformed into the problem of solving above partial differential equation. Here, the transition probability density function with normalization condition

$$\int_{y_{\min}}^{y_{\max}} p(y,t)\mathrm{d}y = 1 \tag{12}$$

where $y_{\max}$ and $y_{\min}$ are the interested maximum and minimum shoreline position, respectively.

The initial condition and boundary condition for equation (11) are shown in the following equations, respectively:

$$\begin{aligned} &p(y,t_0) = p_0(y) \\ &p(y_{\max},t) = 0 \\ &p(y_{\min},t) = 0 \end{aligned} \tag{13}$$

The Fokker-Planck equation is a deterministic partial differential equation for the nonlinear Ito equation problem for which it is difficult to obtain an exact theoretical solution. Since the transfer function ϕ and G are depend on $y(t)$, its partial difference terms would be a complex nonlinear expression, which it is difficult to derive. Various numerical solutions have been found using finite element methods (Spencer and Bergman, 1993, Johnson et al, 1997), finite difference methods (Johnson et al, 1997), and path integral methods (Wehner and Wolfer, 1983). Here a finite difference method may be used to solve equation (11). The backward in time and central in space method is obtained by using a second-order implicit difference approximation.

5 EXAMPLE CALCULATIONS

A simple example will be tested, where a jetty is blocking a littoral drift (Komar, 1983). The jetty being present to the immediate right of cell 100 with a littoral drift arriving from the far left at a rate appropriate for the set wave conditions, $\alpha_0 = 15^0$. Initially the shoreline is straight, corresponding to the *x*-axis (y_i=0), and the model employs the increments $\Delta x = 25$ m and $d_{\mathrm{c}} = 3$ m, $\Delta t = 0.1$ day running for a year. Considering the maximum deposition distance and algorithm precision of the values of the PDF at the nodes, $\Delta y = 0.5$ m, $y_{\min}$=-100m, $y_{\max}$=650m, 1501 nodes has been divided along the cross-shore direction. For all schemes, the following variables will be set expect for specification declare, K_{m} is 0.375 and C_{v} is 0.15 if applicable; mean of y_0 is zero, and deviation of y_0 is 2.5m if applicable. All these random variable confirm to normal distribution.

To consider the uncertainty effects of the input on the accreted process a stochastic procedure may be carried out by FP model. Computed mean of the solution process, $y_1(t)$, is compared with the deterministic solution of equation (1) by numerical scheme (NUME) and analytical scheme (ANALY) based on the mean of K_m is given in Fig. 2.

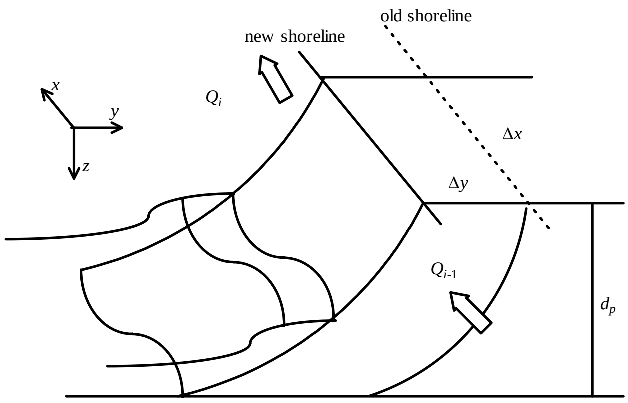


Fig. 1 Definition Sketch

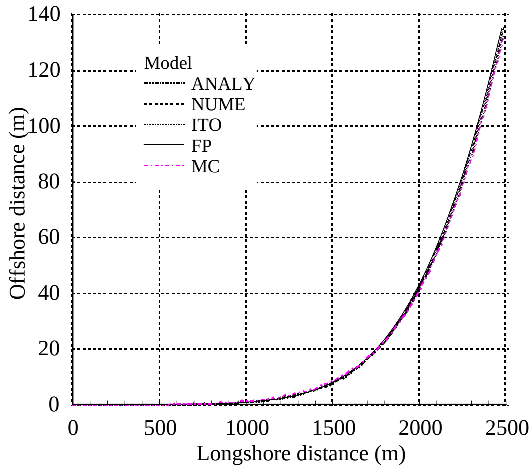


Fig. 2 Comparison of shoreline changing by various model

It shows that, in an average sense, the shoreline position with uncertainties of wave height and boundary condition taken into account differs little from the one obtained using deterministic procedures where the parameters and boundary condition are first averaged. The results on mean shoreline position by some other models, like Ito model (ITO) and direct Monte Carlo model (MC) for equation (7), also shown in Fig. 2. Fig. 2 shows cells after 1 year of the littoral drift being blocked by the jetty using various models. As expected, the cells have built outward to represent the accumulating sand, cell 100 adjacent to the jetty building out the most.

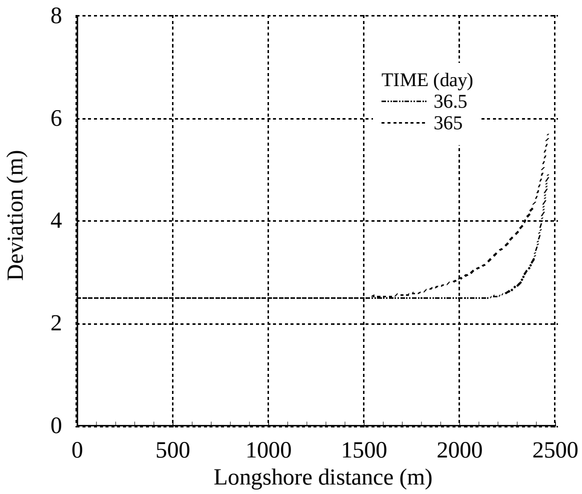


Fig. 3 Distribution of deviation of $y_2(t)$

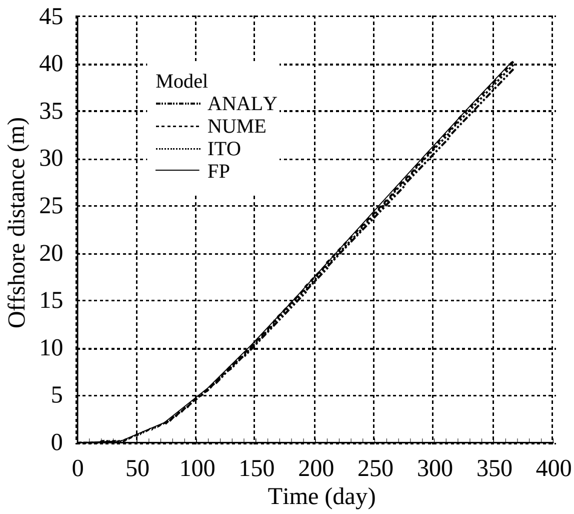


Fig. 4 shoreline position with time on the 80th cell by various model

Fig. 3 shows that the distributions of deviation of shoreline position $y_2(t)$ at 36.5 days and 365 days for all cells. Compared with the results at various steps, its randomness will increase as the increasing of the effect of long-shore transport with time, which behaves a non-linear relation along the shoreline. The deviation is larger adjacent to the jetty than the one far from the jetty, which results from the variation of increment of littoral drift. Fig. 4 shows the progressive accretion through the year at the 80th cell, the shoreline location is quite close to each other by the models. The solution by Fokker-Planck equation is a little bit larger than exact solution, due to the introducing of diffusion.

The variation of $p(y,t)$ with time can be found in Fig. 5 for the 80th cell, incorporating an initial condition of y_0 with a normal distribution. The results show that the accreted shoreline time-variation graph is not longer a deterministic curve, and becomes a stochastic process with some density distributions changing gradually with time. It also can be seen that the density curve becomes wider over time because the process of diffusion, at the same time, the expectation of density curve shifting to a new position owing to accrete process in alongshore direction.

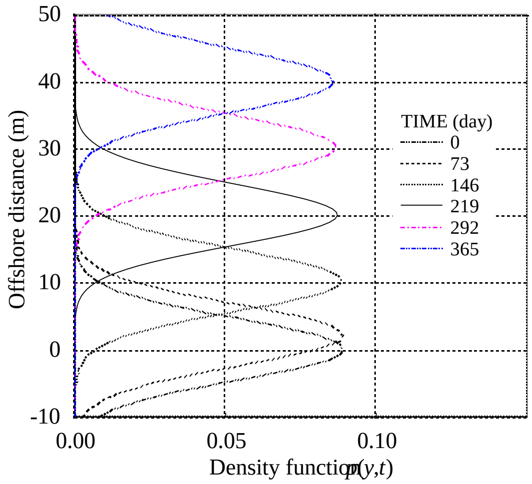


Fig. 5 Variation of PDF’s over time

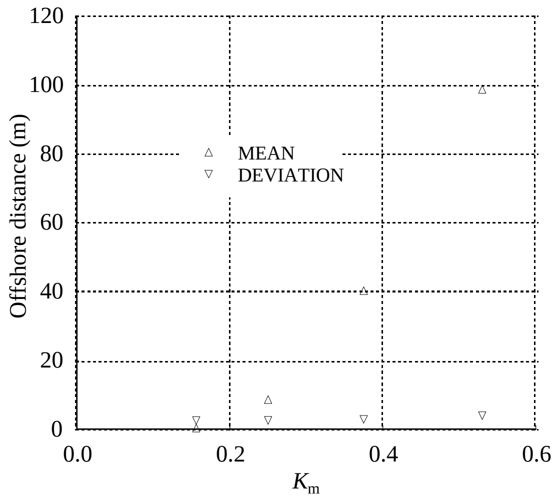
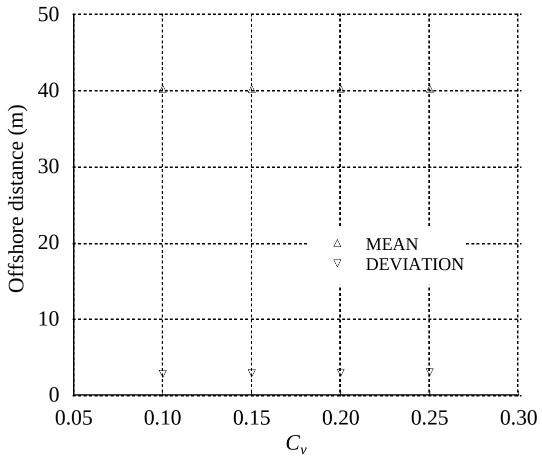


Fig. 6 Mean value and deviation of shoreline position for various K_m

Assume that the variation of K_m and corresponding $y_1(t)$, $y_2(t)$ of the 80th cell at final step are calculated and shown in Fig. 6 by selecting input K_m of 0.156, 0.25, 0.375, and 0.53. Obviously, the expectation of shoreline position increase sharply with the variation of mean of wave height, due to the increasing of wave energy and resulting transport quantities. But a little bit increasing relation on the deviations of shoreline position can be found.

Fig. 7 gives the relation on mean of and deviation of shoreline position with variation coefficient of wave height C_v, whose values are 0.05, 0.15, 0.30 and 0.45, respectively, for the 80th cell at final step. It can be seen that the deviation of shoreline position will increase as the increasing of the randomness parameters of wave height, but the mean of shoreline decrease slightly but no significant effect. These statistic properties can also be found by Monte Carlo model.

The characteristics of $y(t)$ are significantly affected by the standard deviation of y_0 of the initial condition, i.e., initial PDF. The mean and deviation of shoreline position in the 80th cell at final step are shown in Fig. 8 by input a serials deviation of y_0, rang from 0 to 6. The expectation of shoreline position will approach to a stable value as the increasing of deviation of y_0, the reason is that the relative smooth density curve of y_0 will result in the little variation of shoreline because of the smaller diffusion coefficients. It also can be shown that the deviation of shoreline will increase with the increasing of deviation of y_0. By this token, it is quite important to determine the initial deviation of y_0. It should be obtained by historical record data if available.



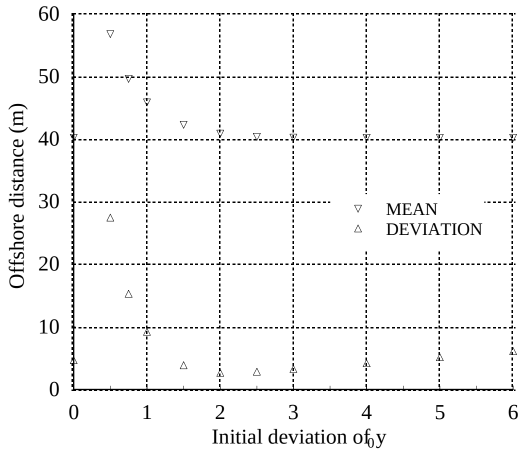


Fig. 7 Variations of mean and deviation of shoreline position with C_v

Fig. 8 Variations of mean and deviation of shoreline position with the initial deviation of y_0

6 CONCLUSIONS

The stochastic differential equation appears to provide a useful mathematical approach to analysing the stochastic phenomena in the shoreline erosion or accretion process and to summarizing the effects of various random factors on the stochastic shoreline position time-variation graph. Through an analysis of the characteristics of Winner process for the sediment drift in the transport process, an ITO equation with a stochastic input term and a random initial condition is derived. Also, by using the Fokker-planck equation which is deterministic parabolic form, the probability density distributions of the shoreline position time-variation graph in transport process are solved.

Comparison the results with some other solutions by deterministic or stochastic approach, very similar tendency and meaningful distribution curves has been gained. It is shown that the proposed random models can reflect the effects of uncertainties of parameters and boundary condition on the shoreline position computations reasonably and can be used for predictive purpose.

The procedure can be used as a powerful tool in conjunction with engineering judgement to evaluate and improve shoreline changing. The results, like density function of $y(t)$, can be easily employed to assess the erosion risk of shoreline by reliability analysis.

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REFERENCE

Komar, P.D. (1983), Handbook of Coastal Processes and Erosion, *CRC Press*.

Dong, P. and Chen, H. (2001), Wave Chronology Effects on Long-Term Shoreline Erosion

Predictions, *Journal of Waterway, Port, Coastal and Ocean Engineering*, 127(3), 186-189.

Reeve D.E. Spivack, M. (2004), Evolution of shoreline position moments, *Coastal Engineering*, 51(8-9), 661-673

Soong, T.T. (1973), Random Differential Equations in Science and Engineering, New York: Academic Press.

Vrijling, J.K., Meijer, G.J. (1992), Probabilistic Coastline Position Computations, *Coastal Engineering*, 17(1-2), 1-23.

Wang, H. F., Anderson, M.A. (1982), Introduction to groundwater modeling: Finite difference and finite element methods. Freeman and Co.

Spencer, Jr., Bergman, L.A. (1993), On the Numerical. Solution of the Fokker-Planck Equation for Nonlinear Stochastic. Systems." *Nonlinear Dynamics* 4, 357-372

Johnson E.A., Wojtkiewicz S.F., Bergman L.A., and Spencer B.F., Jr. (1997), Finite Element and Finite Difference Solutions to the Transient Fokker-Planck Equation, Deutsches Elektronen-Synchrotron (DESY) report DESY-97-161 (A. Bazzani, J. Ellison, H. Mais, and G. Turchetti, eds., DESY, 1998), *Proceedings of a Workshop: Nonlinear and Stochastic Beam Dynamics in Accelerators -- A Challenge to Theoretical and Computational Physics*, Lüneburg, Germany, 29 September 29 – 3 October, pp. 290-306.

Wehner, M.F. ,Wolfer, W.G. (1983), Numerical evaluation of path integral solutions to the Fokker-Planck Equations, *Physics Review*, A27(5):2663-2670.